

HOMOTOPY DECOMPOSITION OF A SUSPENDED REAL TORIC SPACE

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In memory of Professor Samuel Gitler

ABSTRACT. We give p -local homotopy decompositions of the suspensions of real toric spaces for odd primes p . Our decomposition is compatible with the one given by Bahri, Bendersky, Cohen, and Gitler for the suspension of the corresponding real moment-angle complex, or more generally, the polyhedral product. As an application, we obtain a stable rigidity property for real toric spaces. This is a generalized version of the published paper [7].

1. INTRODUCTION

For a simplicial complex K on m -vertices $[m] = \{1, \dots, m\}$ the *real moment-angle complex* $\mathbb{R}\mathcal{Z}_K$ (or *the polyhedral product* $(\underline{D}^1, \underline{S}^0)^K$) of K is defined as follows:

$$\begin{aligned} \mathbb{R}\mathcal{Z}_K &= (\underline{D}^1, \underline{S}^0)^K \\ &:= \bigcup_{\sigma \in K} \{(x_1, \dots, x_m) \in (D^1)^m \mid x_i \in S^0 \text{ when } i \notin \sigma\}, \end{aligned}$$

where $D^1 = [0, 1]$ is the unit interval and $S^0 = \{0, 1\}$ is its boundary. It should be noted that $\mathbb{R}\mathcal{Z}_K$ is a topological manifold if K is a simplicial sphere [3, Lemma 6.13].

There is a canonical \mathbb{F}_2^m -action on $\mathbb{R}\mathcal{Z}_K$ which comes from the \mathbb{F}_2 -action on the pair (D^1, S^0) . Any subgroup of \mathbb{F}_2^m is specified as the kernel of a linear map $\lambda: \mathbb{F}_2^m \rightarrow \mathbb{F}_2^n$ for some $n \leq m$. The quotient space $\mathbb{R}\mathcal{Z}_K / \ker \lambda$ is denoted by $M(K, \lambda)$ and called the *real toric space* associated to K and λ . We denote the i -th column of λ by $\lambda(i)$.

Real toric spaces are a generalization to well-studied classes of spaces. When λ satisfies the following condition, it is called a (mod 2) *characteristic function* of K :

- (1) $\lambda(i_1), \dots, \lambda(i_\ell)$ are linearly independent in \mathbb{F}_2^n if $\{i_1, \dots, i_\ell\} \in K$.

Lemma 1.1. The action of $\ker \lambda$ on $\mathbb{R}\mathcal{Z}_K$ is free if and only if (1) is satisfied.

Proof. Let $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}\mathcal{Z}_K = (\underline{D}^1, \underline{S}^0)^K$ be the fixed point of an element $g = (g_1, g_2, \dots, g_m)^t \in \ker \lambda \subset \mathbb{F}_2^m$. This means either $g_i = 0$ or $x_i \in (D^1)^{\mathbb{F}_2} = \{1/2\}$ for all $i \in [m]$. Let $\sigma \in K$ be a simplex such that $x \in (\underline{D}^1, \underline{S}^0)^\sigma$ and λ_σ be the sub-matrix of λ consisting of columns corresponding to σ . Let g_σ be the sub-vector of g corresponding to σ . Since $g \in \ker \lambda$, we have

$$\lambda g = \lambda_\sigma g_\sigma + \lambda_{[m] \setminus \sigma} g_{[m] \setminus \sigma} = 0.$$

Since \mathbb{F}_2 acts on S^0 freely, we have $g_i = 0$ for $i \notin \sigma$. Then, by the previous equation we have $\lambda_\sigma g_\sigma = 0$. Therefore, the condition (1) implies $g_\sigma = 0$ and $g = 0$. On the other hand, if (1)

does not hold for a simplex $\sigma \in K$, the point $x_i = \begin{cases} 1/2 & (i \in \sigma) \\ 0 & (i \notin \sigma) \end{cases}$ is fixed by an element in $\ker \lambda$. □

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For a characteristic function λ , the corresponding real toric space $M(K, \lambda)$ is known as a *small cover* [10] if K is a polytopal $(n-1)$ -sphere, and as a *real topological toric manifold* [12] if K is a star-shaped $(n-1)$ -sphere.

In [1, Theorem 2.21] it is shown that there is a homotopy equivalence

$$(2) \quad \Sigma \mathbb{R} \mathcal{Z}_K \simeq \Sigma \bigvee_{I \in [m]} \Sigma |K_I| \simeq \Sigma \bigvee_{I \notin K} \Sigma |K_I|,$$

where K_I is the full subcomplex of K on the vertex set I and $|K_I|$ is its geometric realization so that $|K_I|$ is contractible when $I \in K$. In this short note, we give an analogous odd primary decomposition of the suspension of $M(K, \lambda)$.

Theorem 1.2. Let $M(K, \lambda)$ be a real toric space. Localized at an odd prime p or the rationals (denoted by $p = 0$) there is a homotopy equivalence

$$\Sigma(M(K, \lambda)) \simeq_p \Sigma \bigvee_{I \in \text{Row}(\lambda)} \Sigma |K_I|,$$

where $\text{Row}(\lambda)$ is the space of m -dimensional \mathbb{F}_2 -vectors spanned by the rows of λ associated to λ . We identify vectors in \mathbb{F}_2^m with subsets of $[m]$ in the obvious manner.

Remark 1.3. When λ is the identity matrix, $M(K, \lambda)$ is nothing but $\mathbb{R} \mathcal{Z}_K$. On the other hand, when λ is zero, $M(K, \lambda)$ is the cone over $|K|$ and contractible.

The restriction to odd primes arises because when $|\ker \lambda|$ is inverted in a coefficient ring R then the quotient map $\mathbb{R} \mathcal{Z}_K \rightarrow M(K, \lambda)$ induces an injection in cohomology with image the invariant subring $H^*(\mathbb{R} \mathcal{Z}_K; R)^{\ker \lambda}$ (see, for example, [2, §III.2]). This will be used to help analyze the topology of $M(K, \lambda)$. As $\ker \lambda$ has order a power of 2 we can take R to be $\mathbb{Z}_{(p)}$ or \mathbb{Q} . In fact, Theorem 1.2 fails when $p = 2$ in simple cases. For example, if K is the boundary of a triangle and $\lambda = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, then $M(K, \lambda) = \mathbb{R} P^2$ but each $\Sigma |K_I|$ is contractible.

Recent work of Yu [14] gave a different decomposition of the suspension of certain quotient spaces of $\mathbb{R} \mathcal{Z}_K$. He considers λ which is associated to a partition on the vertices of K , and proves that $M(K, \lambda)$ decomposes analogously to the Bahri, Bendersky, Cohen and Gitler decomposition. Yu's decomposition has the advantage of working integrally, but it has the disadvantage of working only for particular λ . Our decomposition, by contrast, works only after localizing at an odd prime but holds for any λ .

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2. POLYHEDRAL PRODUCT AND ITS STABLE DECOMPOSITION

Let us first recall Bahri, Bendersky, Cohen and Gitler's argument in [1]. To make it more clear, we present it in its full polyhedral product form. Let K be a simplicial complex on the vertex set $[m]$ and for $1 \leq i \leq m$ let (X_i, A_i) be pairs of pointed CW -complexes. If σ is a face of K let

$$(\underline{X}, \underline{A})^\sigma = \prod_{i=1}^m Y_i \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma \end{cases}.$$

The *polyhedral product* is

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma.$$

Notice that $(\underline{X}, \underline{A})^K$ is a subspace of the product $\prod_{i=1}^m X_i$. There is a canonical quotient map from the product to the smash product, $\prod_{i=1}^m X_i \rightarrow \bigwedge_{i=1}^m X_i$. The *smash polyhedral product*

$(\widehat{\underline{X}, \underline{A}})^K$ is the image of the composite $(\underline{X}, \underline{A})^K \rightarrow \prod_{i=1}^m X_i \rightarrow \bigwedge_{i=1}^m X_i$. In particular, mapping onto the image gives a map $(\underline{X}, \underline{A})^K \rightarrow (\widehat{\underline{X}, \underline{A}})^K$.

Let $I \subset [m]$. As in [11, 2.2.3(i)], projecting $\prod_{i=1}^m X_i$ onto $\prod_{i \in I} X_i$ induces a map of polyhedral products $(\underline{X}, \underline{A})^K \rightarrow (\underline{X}, \underline{A})^{K_I}$. We then obtain a composition into a smash polyhedral product:

$$p_I : (\underline{X}, \underline{A})^K \rightarrow (\underline{X}, \underline{A})^{K_I} \rightarrow (\widehat{\underline{X}, \underline{A}})^{K_I}.$$

Suspending, we can add every such composition over all full subcomplexes of K , giving a composition

$$\overline{H} : \Sigma(\underline{X}, \underline{A})^K \xrightarrow{\text{comul}} \bigvee_{I \subset [m]} \Sigma(\underline{X}, \underline{A})^{K_I} \xrightarrow{\vee \Sigma p_I} \bigvee_{I \subset [m]} \Sigma(\widehat{\underline{X}, \underline{A}})^{K_I}.$$

Bahri, Bendersky, Cohen and Gitler [1, Theorem 2.10] show that \overline{H} is a homotopy equivalence.

Further, in the special case when each X_i is contractible, they show that there is a homotopy equivalence $(\widehat{\underline{X}, \underline{A}})^{K_I} \simeq \Sigma(|K_I| \wedge \widehat{A}^I)$ [1, Theorem 2.19], where $\widehat{A}^I = \bigwedge_{j=1}^k A_{i_j}$ for $I = (i_1, \dots, i_k)$. Consequently, when each X_i is contractible the map \overline{H} specializes to a homotopy equivalence

$$H : \Sigma(\underline{X}, \underline{A})^K \rightarrow \bigvee_{I \subset [m]} \Sigma(\underline{X}, \underline{A})^{K_I} \rightarrow \bigvee_{I \subset [m]} \Sigma(\widehat{\underline{X}, \underline{A}})^{K_I} \xrightarrow{\simeq} \bigvee_{I \subset [m]} \Sigma^2(|K_I| \wedge \widehat{A}^I).$$

In our case, each pair (X_i, A_i) equals (D^1, S^0) and D^1 is contractible. As there is a homotopy equivalence $S^0 \wedge S^0 \simeq S^0$, each \widehat{A}^I is homotopy equivalent to S^0 . Therefore there are homotopy equivalences

$$(3) \quad \widehat{\mathbb{RZ}}_{K_I} := (\widehat{D^1, S^0})^{K_I} \xrightarrow{\simeq} \Sigma|K_I| \wedge \widehat{A}^I \simeq \Sigma|K_I| \wedge S^0 \simeq \Sigma|K_I|.$$

Thus the map H becomes a homotopy equivalence

$$H : \Sigma \mathbb{RZ}_K \rightarrow \bigvee_{I \subset [m]} \Sigma \mathbb{RZ}_{K_I} \rightarrow \bigvee_{I \subset [m]} \Sigma \widehat{\mathbb{RZ}}_{K_I} \xrightarrow{\simeq} \bigvee_{I \subset [m]} \Sigma^2|K_I|.$$

It is in this form that we will use the Bahri, Bendersky, Cohen and Gitler decomposition because, as we will see shortly, it corresponds to a module decomposition of a differential graded algebra R_K whose cohomology equals $H^*(\mathbb{RZ}_K)$.

3. PROOF OF THE MAIN THEOREM

Consider the following diagram

$$(4) \quad \begin{array}{ccccc} \Sigma \mathbb{RZ}_K & \xrightarrow{\bar{H}} & \Sigma \bigvee_{I \subset [m]} \widehat{\mathbb{RZ}}_{K_I} & \xrightarrow{\simeq} & \Sigma \bigvee_{I \subset [m]} \Sigma|K_I| \\ \downarrow \Sigma q & & \uparrow \Sigma \text{incl} & & \\ \Sigma M(K, \lambda) & \xleftarrow{\phi} & \Sigma \bigvee_{I \in \text{Row}(\lambda)} \widehat{\mathbb{RZ}}_{K_I} & \xrightarrow{\simeq} & \Sigma \bigvee_{I \in \text{Row}(\lambda)} \Sigma|K_I| \end{array}$$

where, by definition, $\phi = \Sigma q \circ \bar{H}^{-1} \circ \Sigma \text{incl}$.

To prove Theorem 1.2 we will show that ϕ^* induces an isomorphism on cohomology. From now on, assume that coefficients in cohomology are \mathbb{Q} or $\mathbb{Z}_{(p)}$, where p is an odd prime.

First, by [5, Theorem 5.1] the cohomology ring of \mathbb{RZ}_K is given as follows. Let $\mathbb{Z}_{(p)}\langle u_1, \dots, u_m, t_1, \dots, t_m \rangle$ be the free associative algebra over the indeterminants of $\deg u_i = 1, \deg t_i = 0$ ($i = 1, \dots, m$). Define a differential graded algebra R_K as the quotient

$$R_K = \frac{\mathbb{Z}_{(p)}\langle u_1, \dots, u_m, t_1, \dots, t_m \rangle}{(u_i^2, u_i u_j + u_j u_i, u_i t_i - u_i, t_i u_i, t_i u_j - u_j t_i, t_i^2 - t_i, t_i t_j - t_j t_i, u_\sigma \ (\sigma \notin K))}$$

where $i \neq j$ and $d(t_i) = u_i$ for each $i = 1, \dots, m$. Then $H^*(\mathbb{RZ}_K) \simeq H^*(R_K)$. We shall use the notation u_σ (respectively, t_σ) for the monomial $u_{i_1} \cdots u_{i_k}$ (respectively, $t_{i_1} \cdots t_{i_k}$) where

$\sigma = \{i_1, \dots, i_k\}$, $i_1 < \dots < i_k$, is a subset of $[m]$. For $I \subset [m]$, denote by R_{K_I} the differential graded sub-module of R_K spanned by the monomials $\{u_\sigma t_{I \setminus \sigma} \mid \sigma \in K_I\}$. Observe from the definitions of R_K and R_{K_I} that there is an additive isomorphism $R_K = \bigoplus_{I \subset [m]} R_{K_I}$.

Lemma 3.1. There is an additive isomorphism

$$H^*(R_{K_I}) \simeq \tilde{H}^*(\widehat{\mathbb{R}\mathcal{Z}_{K_I}})$$

and the projection $p_I: \mathbb{R}\mathcal{Z}_K \rightarrow \widehat{\mathbb{R}\mathcal{Z}_{K_I}}$ induces the inclusion $p_I^*: H^*(R_{K_I}) \hookrightarrow H^*(R_K)$.

Proof. The first assertion follows from $\widehat{\mathbb{R}\mathcal{Z}_{K_I}} \simeq \Sigma|K_I|$ (see (3)) and the isomorphism $H^*(R_{K_I}) \simeq \tilde{H}^{*-1}(|K_I|)$ given by

$$\begin{aligned} R_{K_I} &\rightarrow C^*(K_I) \\ u_\sigma t_{I \setminus \sigma} &\mapsto \sigma^*, \end{aligned}$$

where $C^*(K_I)$ is the simplicial cochain complex of K_I ([5, Proposition 3.3]).

To show the second assertion, we look more closely at the isomorphism $H^*(R_K) \simeq H^*(\mathbb{R}\mathcal{Z}_K)$. From [5, §3.2], the monomials $u_\sigma t_{I \setminus \sigma}$ are mapped into the image of $p_I^*: C_e^*(\widehat{\mathbb{R}\mathcal{Z}_{K_I}}) \rightarrow C_e^*(\mathbb{R}\mathcal{Z}_K)$, where C_e^* denotes the cellular cochain complex. By combining this with the first assertion, we deduce the second assertion. \square

Now we investigate the maps appearing in (4). Since $|\ker \lambda|$ is a unit in the coefficient ring $\mathbb{Z}_{(p)}$, the map q^* is injective with image $H^*(\mathbb{R}\mathcal{Z}_K)^{\ker \lambda}$. Notice that in cohomology $incl$ induces the projection $incl^*: \bigoplus_{I \subset [m]} H^*(R_{K_I}) \rightarrow \bigoplus_{I \in \text{Row}(\lambda)} H^*(R_{K_I})$. Recall that $\bar{H} = \Sigma \bigvee_{I \subset [m]} p_I \circ comul$ and $\phi = \Sigma q \circ \bar{H}^{-1} \circ \Sigma incl$. So ϕ^* is the composite

$$\phi^*: H^*(\Sigma M(K, \lambda)) \simeq H^*(\Sigma \mathbb{R}\mathcal{Z}_K)^{\ker \lambda} \hookrightarrow H^*(\Sigma \mathbb{R}\mathcal{Z}_K) \xrightarrow{\simeq} \bigoplus_{I \subset [m]} H^*(\Sigma R_{K_I}) \rightarrow \bigoplus_{I \in \text{Row}(\lambda)} H^*(\Sigma R_{K_I}),$$

where Σ for graded modules means the degree shift in the positive degree parts.

We aim to show that ϕ^* is an isomorphism. To see this, we need two lemmas.

Lemma 3.2 ([9, Section 4]). The Reynolds operator

$$N(x) := \frac{1}{|\ker \lambda|} \sum_{g \in \ker \lambda} gx$$

induces an additive isomorphism $\bigoplus_{I \in \text{Row}(\lambda)} R_{K_I} \xrightarrow{\simeq} R_K^{\ker \lambda}$, where $R_K^{\ker \lambda}$ is the $\ker \lambda$ -invariant ring of R_K . Furthermore, for a monomial $x = u_\sigma t_{I \setminus \sigma}$, $N(x)$ has the unique maximal term x , where the order is given by the containment of the index set. \square

Lemma 3.3. The composite

$$\Phi: R_K^{\ker \lambda} \hookrightarrow R_K \simeq \bigoplus_{I \subset [m]} R_{K_I} \xrightarrow{\pi} \bigoplus_{I \in \text{Row}(\lambda)} R_{K_I}$$

is an isomorphism of rings, where π is the projection. Note that we consider the ring structure of $\bigoplus_{I \in \text{Row}(\lambda)} R_{K_I}$ induced by that of R_K .

Proof. We first show this is surjective. Take an element $x \in \bigoplus_{I \in \text{Row}(\lambda)} R_{K_I}$. We induct on the size of the index set of x . By Lemma 3.2, the terms in $\pi(N(x) - x) \in \bigoplus_{I \in \text{Row}(\lambda)} R_{K_I}$ has an index set strictly smaller than that for x . By induction hypothesis, there is an element $y \in R_K^{\ker \lambda}$ such that $\Phi(y) = \pi(N(x) - x)$. Put $z = N(x) - y \in R_K^{\ker \lambda}$ and we have $\Phi(z) = \pi(N(x)) - \Phi(y) = \pi(x) = x$.

On the other hand, suppose $\Phi(y) = 0$ for some $y \in R_K^{\ker \lambda}$. By Lemma 3.2, there is $x \in \bigoplus_{I \in \text{Row}(\lambda)} R_{K_I}$ such that $y = N(x)$ and y must contain the maximal terms in x . Thus, $\Phi(y) = 0$ implies $x = 0$ and $y = N(x) = 0$. \square

Proof of Theorem 1.2. Since $H^*(\mathbb{R}\mathcal{Z}_K)^{\ker \lambda} \simeq H^*(R_K^{\ker \lambda})$, the definitions of ϕ and Φ imply that $\Sigma\Phi^* = \phi^*$. Therefore, by Lemma 3.3, ϕ^* is an isomorphism. \square

Corollary 3.4. We have the ring isomorphism

$$H^*(M(K, \lambda)) \simeq H^*\left(\bigoplus_{I \in \text{Row}(\lambda)} R_{K_I}\right),$$

where the ring structure of the right hand side is induced by that of R_K .

4. STABLE RIGIDITY OF REAL TORIC SPACES

In this section, we give an application of Theorem 1.2 to a stable rigidity property of real toric spaces.

Corollary 4.1. Let $M(K, \lambda)$ be a real toric space over K . When $|K_I|$ for any $I \in \text{Row}(\lambda)$ suspends to a wedge of spheres after localization at an odd prime p , $\Sigma M(K, \lambda)$ is homotopy equivalent to a wedge of spheres after localization at p . Let $N(K', \mu)$ be another real toric spaces over K' , where $|K'_I|$ for any $I \in \text{Row}(\mu)$ suspends to a wedge of spheres after localization at p . Then, if $H^*(M(K, \lambda); \mathbb{F}_p) \simeq H^*(N(K', \mu); \mathbb{F}_p)$ as modules, we have $\Sigma M(K, \lambda) \simeq_p \Sigma N(K', \mu)$. \square

Real toric spaces associated to graphs. Given a connected simple graph G with $n + 1$ nodes $[n + 1]$, the *graph associahedron* P_G ([6]) of dimension n is a convex polytope whose facets correspond to the connected subgraphs of G . Let K be the boundary complex of P_G . We can describe K directly from G : the vertex set of K consists proper subsets $T \subsetneq [n + 1]$ such that $G|_T$ are connected and the simplices are the tubings of G . We define a mod 2 characteristic map λ_G on K as follows:

$$\lambda_G(T) = \begin{cases} \sum_{t \in T} \mathbf{e}_t, & \text{if } n + 1 \notin T; \\ \sum_{t \notin T} \mathbf{e}_t, & \text{if } n + 1 \in T, \end{cases}$$

where \mathbf{e}_t is the t -th coordinate vector of \mathbb{F}_2^n . Then we have a real toric manifold $M(G) := M(K, \lambda_G)$ associated to G .

The *signed a -number* $sa(G)$ of G is defined recursively by

$$sa(G) = \begin{cases} 1, & \text{if } G = \emptyset; \\ 0, & \text{if } G \text{ has a connected component with odd number of nodes} \\ -\sum_{T \subsetneq [n+1]} sa(G|_T), & \text{otherwise,} \end{cases}$$

and the *a -number* $a(G)$ of G is the absolute value of $sa(G)$. As shown in [8], there is a bijection φ from $\text{Row}(\lambda_G)$ to the set of subgraphs of G having an even number nodes and $|K_I|$ for $I \in \text{Row}(\lambda_G)$ is homotopy equivalent to $\bigvee^{a(\varphi(I))} S^{|\varphi(I)|/2-1}$ where $|\varphi(I)|$ is the number of nodes of $\varphi(I)$. By Theorem 1.2 we obtain the following.

Corollary 4.2. We have a homotopy equivalence

$$\Sigma M(G) \simeq_p \bigvee_{I \in \text{Row}(\lambda_G)} \bigvee^{a(\varphi(I))} S^{|\varphi(I)|/2+1} \quad \text{for any odd prime } p.$$

\square

Now, we define the *a_i -number* $a_i(G)$ of G by

$$a_i(G) = \sum_{\substack{T \subsetneq [n+1] \\ |T|=2i}} a(G|_T).$$

Then, $a_i(G)$ coincides the i -th Betti number $\beta^i(M(G); \mathbb{F}_p)$ of $M(G)$. It should be noted that, by Corollary 4.2, if two graphs G_1 and G_2 have the same a_i -numbers for all i 's, then $\Sigma M(G_1) \simeq_p \Sigma M(G_2)$ for any odd prime p .

Example 4.3. Let P_4 be a path graph of length 3, and $K_{1,3}$ a tree with one internal node and 3 leaves (known as a *claw*). One can compute $a_i(G) := \sum_{\substack{T \subseteq [n+1] \\ |T|=2i}} a(G|_T)$ as follows:

$$\begin{aligned} a_0(P_4) &= a_0(K_{1,3}) = 1 \\ a_1(P_4) &= a_1(K_{1,3}) = 3 \\ a_2(P_4) &= a_2(K_{1,3}) = 2 \\ a_i(P_4) &= a_i(K_{1,3}) = 0 \quad \text{for } i > 2. \end{aligned}$$

Hence, by Corollary 4.2, $\Sigma M(P_4) \simeq_p \Sigma M(K_{1,3})$ for any odd prime p although $\Sigma M(P_4)$ and $\Sigma M(K_{1,3})$ are not homotopy equivalent since they have different mod-2 cohomology.

Real toric spaces over fillable complexes. There is a wide class of simplicial complexes on which every real toric space satisfies the assumption in Corollary 4.1.

Definition 4.4 ([13, Definition 7.6]). Let K be a simplicial complex. Let K_1, \dots, K_s be the connected components of K , and let \hat{K}_i be a simplicial complex obtained from K_i by adding all of its minimal non-faces. Then K is said to be \mathbb{F}_p -homology fillable if (1) for each i there are minimal non-faces M_1^i, \dots, M_r^i of K such that $K_i \cup M_1^i \cup \dots \cup M_r^i$ is acyclic over \mathbb{F}_p , and (2) \hat{K}_i is simply connected for each i .

Moreover, we say that K is *totally* \mathbb{F}_p -homology fillable when K_I is \mathbb{F}_p -homology fillable for any $\emptyset \neq I \subset [m]$.

Proposition 4.5 ([13, Proposition 7.11]). If K is \mathbb{F}_p -homology fillable, then $(\Sigma|K|)_{(p)}$ is a wedge of p -local spheres.

There is a large class of simplicial complexes which are totally homology fillable.

Proposition 4.6 ([13, Theorem 8.21]). If the Alexander dual of K is sequentially Cohen-Macaulay over \mathbb{F}_p ([4]), then K is totally \mathbb{F}_p -homology fillable.

Note that shifted and shellable simplicial complexes are sequentially Cohen-Macaulay over \mathbb{F}_p .

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